The Wicked Awesome Central Place Light Show

A Command Performance

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Intoduction

Hexagonia[1] is an imaginary country whose urban areas are shaped and arranged hexagonally, like bathroom tiles. It is also the name of an electronic display that dynamically visualizes the geometry of cities in this theoretical landscape. Cities of this fanciful land are embodied by pulsating lights, which together simulate a system of 327 cities of varying size. Hexagonia's circuits orchestrate neon glow lamps to pulse at particular rates and luminosities at a combined rate of nearly 400 pulses per second. The circuits are simple analog *finite-state machines*[2]. All together, they have a very large number of possible states, each of which displays as a unique but transient visual configuration. This qualifies Hexagonia as a computer graphics device, built at a time when no digital computer was able to create fast-moving visualizations like it displays.

The author conceived of and constructed Hexagonia in 1969 to satisfy a theoretical geography course requirement at Harvard's Graduate School of Design, and was allowed to keep it. As it was known then, *Steady State Microcosm* earned an A from the instructor, geographer Michael Woldenberg. In the decades since I built Hexagonia, it has graced the walls of some of my apartments and slept in the basement of others. I am fortunate to still have the device after moving many times and that it still works. Given that the machine is likely to outlast me, and lest my memory fail, I decided to document my youthful folly herein.

The Hexagonia light machine is shown below, suspended by wires on a wall. It measures 38 in (96.5 cm) wide by 32.5 in (82.5 cm) high by 5.5 (14 cm) in deep. The wire at the bottom connects it to an AC power outlet.

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[1] This name was recently coined by the author's daughter, Deniz Dutton. The device was originally called *Steady State Microcosm*.

[2] A finite state machine is any device that has a defined status at a given time and can change its status in response to physical or logical input events (undergoes state transitions). Hexagonia is an ensemble of binary finite state machines that taken together have a very large repertoire of distinct configurations.

Figure 1: The Hexagonia finite-state machine analog display device. Photo by author, 2015.

You may be wondering what could have prompted a graduate student in city planning to spend several hundred hours and as many dollars fabricating a light show in lieu of a paper that might have taken 10 or 15 hours to write. Most likely, it came from an urgent impulse to create works of art that embody timeless mathematical principles, in space and in real time—a mission to bring art to science and science to art. Too, it was the psychedelic 1960s and the display promised to be quite a trip, as we liked to say then.

Central Place Geography

The geographical model that Hexagonia illustrates is about 80 years old, but derives from an even earlier theory by German economist Johann Heinrich von Thünen (1783–1850), who believed it was possible to optimize the productive use of land. Given, say, farm plots and woodlots near a town, he tried to show that landowners would choose crops that maximized their profit per unit of land (income – (costs for production + transport to market)). In his 1826 text *The Isolated State*, von Thünen postulated that different productive activities should occupy concentric rings around market centers, optimizing the use and price of land. In theory, each type of activity will occupy the ring that minimizes costs and maximizes returns for its owners and optimally benefit the economy as a whole. (Besides being an economist, von Thünen was a landowner.)

About a century later, German geographer Walter Christaller (1893–1969) elaborated von Thünen's approach to formulate an urban location model he called Central Place Theory. Christaller's work is important because it laid a foundation for studying cities as urban and regional systems, rather than as unique isolated entities to be studied and compared. Central Place Theory has been incorporated in subsequent models of regional structure and has also informed studies of natural hierarchical systems.

In a nutshell, Central Place Theory posits that consumers and producers are distributed uniformly on a plane (a region containing cities and towns of various sizes). A minimum number of consumers are needed to sustain each type of economic activity. Larger cities support more economic functions and more specializations than do smaller cities. Consumers visit the cities closest to them that satisfy particular economic needs. Given this idealized economic landscape, the theory predicts that cities will be arrayed in a triangular grid, giving each city a hexagonal region of influence (its hinterland). Cities with more functions and activity command larger regions than those with fewer, and their hinterlands overlap those of smaller cities in their neighborhood.

In such a central place system, each city has a circular von Thünen hinterland that abuts those of its six similarly sized neighbors along straight boundaries to create equally sized hexagonal regions that fully tile the plane. Cities are compact and occupy the central points of their hexagonal hinterlands. Those hinterlands are similarly partitioned by the next smaller set of cities, and so on, down to the level of villages. Each hierarchical level has more cities than the ones above it, spaced closer together. [3]

Central Place Geometry

A hierarchy of hexagons can have 3, 4 or 7 lower-level hexagons for each one at a given level, depending on whether the smaller hexagons are centered at vertices, edges, or faces of the larger level's network, respectively. Christaller argued that these three configuration types respectively optimize "marketing," "transport," and "administration" functions. The "k=3" hierarchy used in Hexagonia illustrates the first of these, his "marketing principle." It minimizes the number of towns needed to serve a given region, but by being more widely spaced they entail higher transport costs. Hexagonia uses it because it shows the nesting of regions within regions most clearly. Figure 2 shows 3 orders of central places in a k=3 hierarchy.

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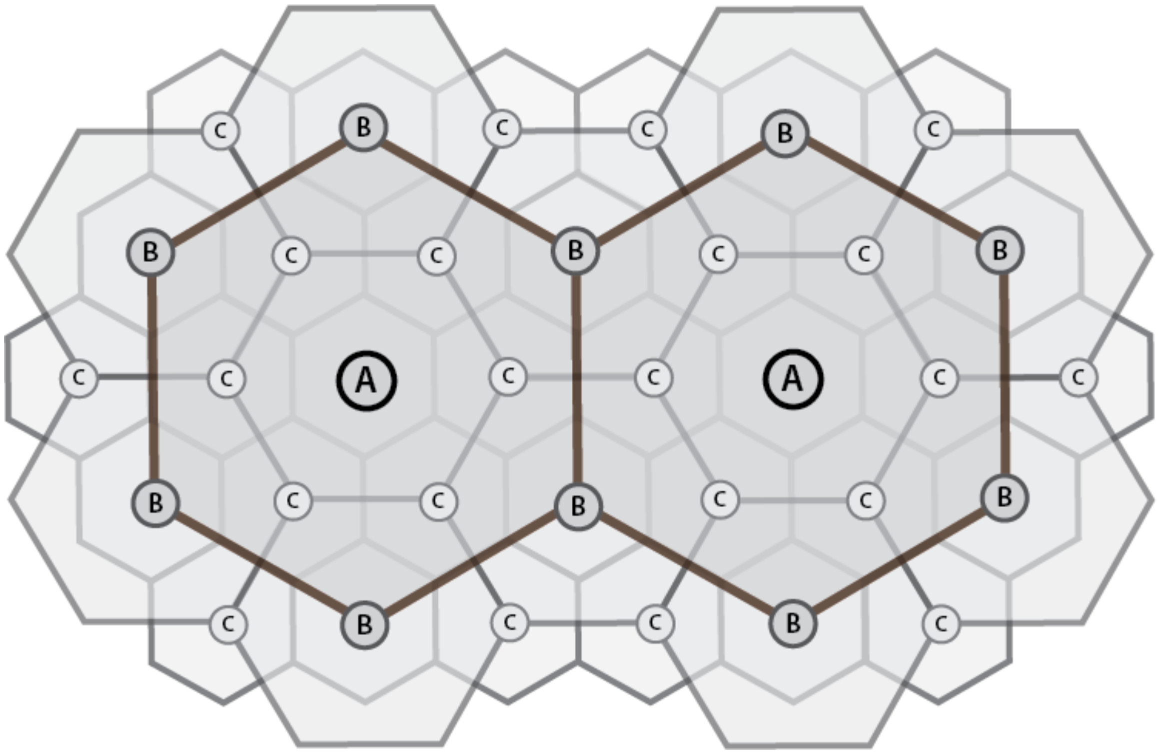
[3] Christaller conceptualized central places top down, i.e. by assuming a dominant city and then allocating successively smaller ones with fewer functions or ones having more restricted radii. German economic geographer August Lösch later upended Christaller's theory by overlaying higher-order places on a uniform landscape of villages. His approach yields many possible central place hierarchies, including the ones (K=3, 4, 7) that Christaller had posited. In the physical world, Löschian systems of cities are harder to identify than Christallerian ones.

Figure 2: Fragment of a k = 3 (area ratio) hexagonal central place hierarchy showing two top-level cities (A), 10 second-level cities (B), and 22 third-level cities (C).

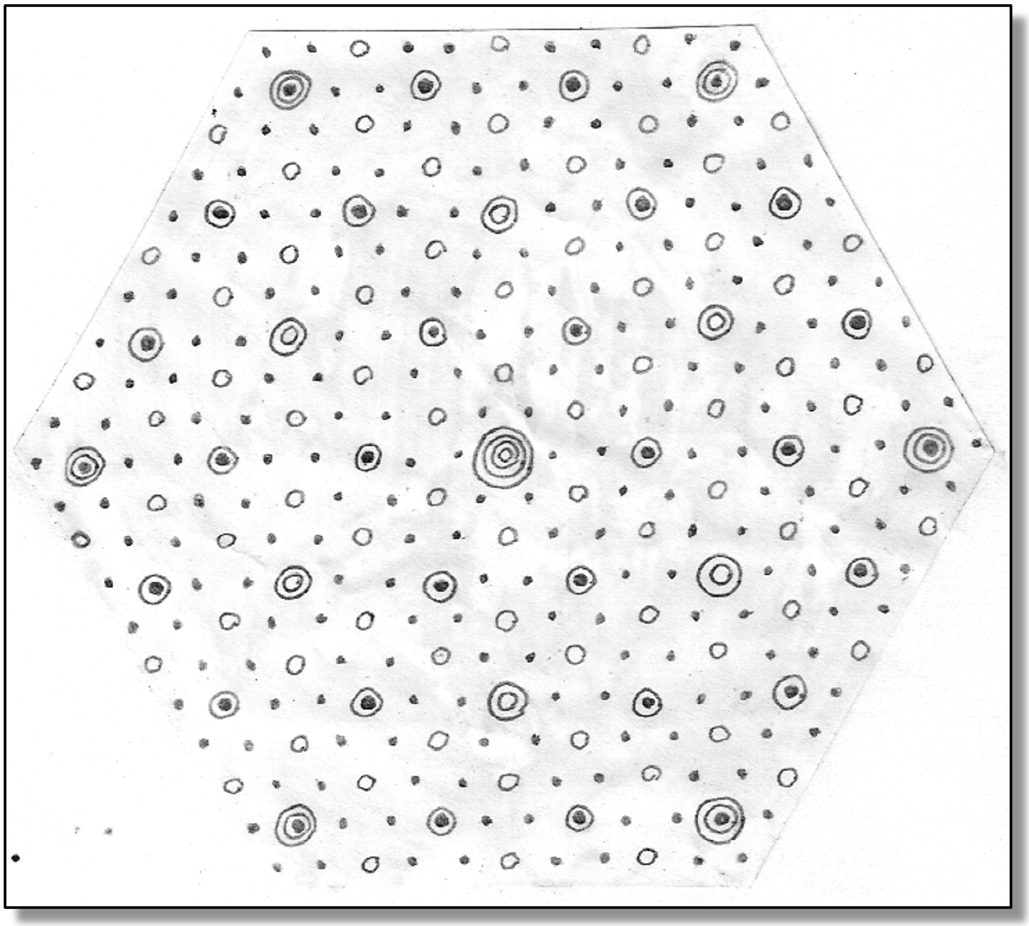


Why hexagons, you might wonder, and not squares? After all, a square can easily be partitioned into 4, 9, or 16 smaller squares, each with the same shape and fully enclosed by its parent, and those squares can be similarly subdivided into smaller units. However, a triangular grid minimizes separation between neighbors and makes those distances more uniform. Hexagons naturally arise from a triangular lattice, but their children are less well behaved, as most of them include areas not covered by their parent. And, as a result of not properly nesting, their subdivisions have more complicated shapes.

Being more nearly circular than squares, tiling hexagons puts their central points closer to each other than those of square regions. Also, instead of four vectors connecting a given city with its neighbors, there are six, which more closely approximates how actual political units connect to their neighbors. (If you inspect a map of boundaries of administrative regions, such as the states of the US, you see many more units that abut six neighbors than touch four.)

Figure 3, a pencil drawing from the original unpublished documentation, shows the layout of the lamps. High-order places have more and larger circles. Note the most central place, which commands the entire region. If you look at the still frame in Figure 6, you will not see it, although it should be visually most prominent. It was, but once the device was finished and fired up, the central lamp's high brightness and rapid pulsing (20 flashes per second) seemed distracting, and so the author masked it to enable the viewer's eye to wander more easily.

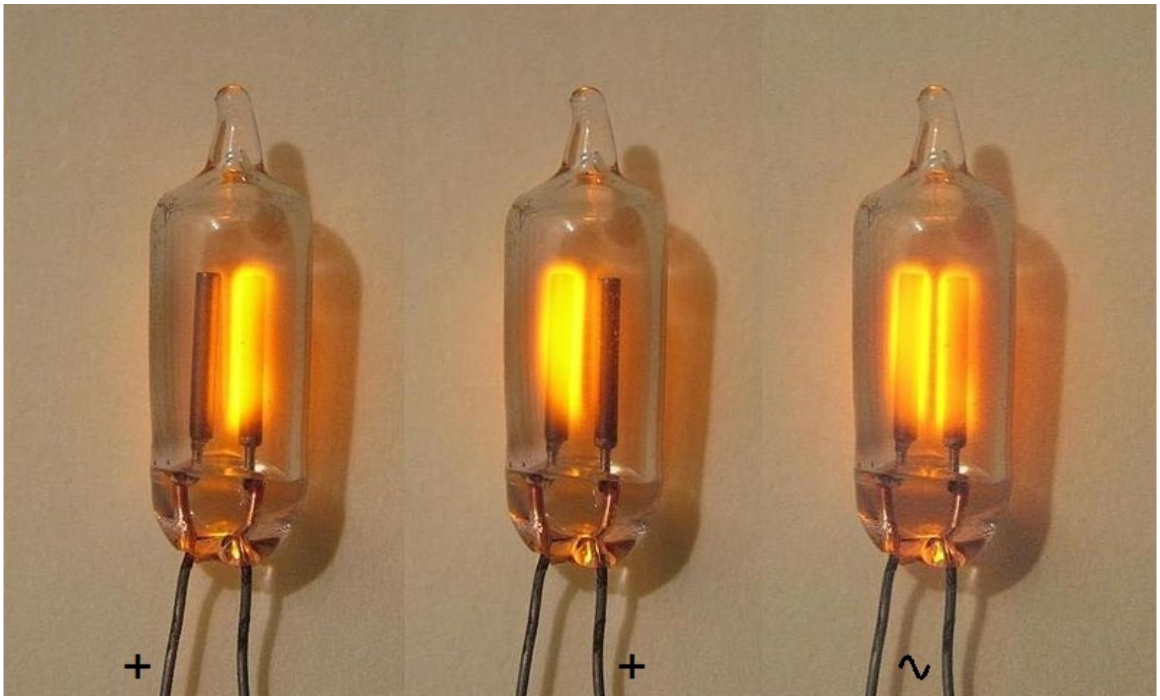
Figure 3: Layout of the Hexagonia display panel showing its k=3 hierarchy of places. Each symbol represents one NE-2 neon lamp



Central Place Graphics

As Figure 1 shows, the device is housed in a hexagonal wooden frame 31 inches wide by 8 inches deep, occupied by a white Plexiglas diffuser. Inside the frame are 164 analog circuits called relaxation oscillators, also known as multivibrators. (When you see a car flashing a turn signal, that's a relaxation oscillator at work.) Running independently in parallel, each circuit alternately blinks two 0.4 milliamp NE-2 neon gas discharge lamps (see figure 4) at a specific rate and luminosity. Circuits are powered by 120 VAC house current via an isolation transformer and converted to DC by a silicon rectifier. Total power consumption is on the order of 10 Watts.

Figure 4: NE-2 Glow Lamps operating on direct current with different polarity (left two bulbs) and alternating current (rightmost bulb). Photo from wikimedia.org

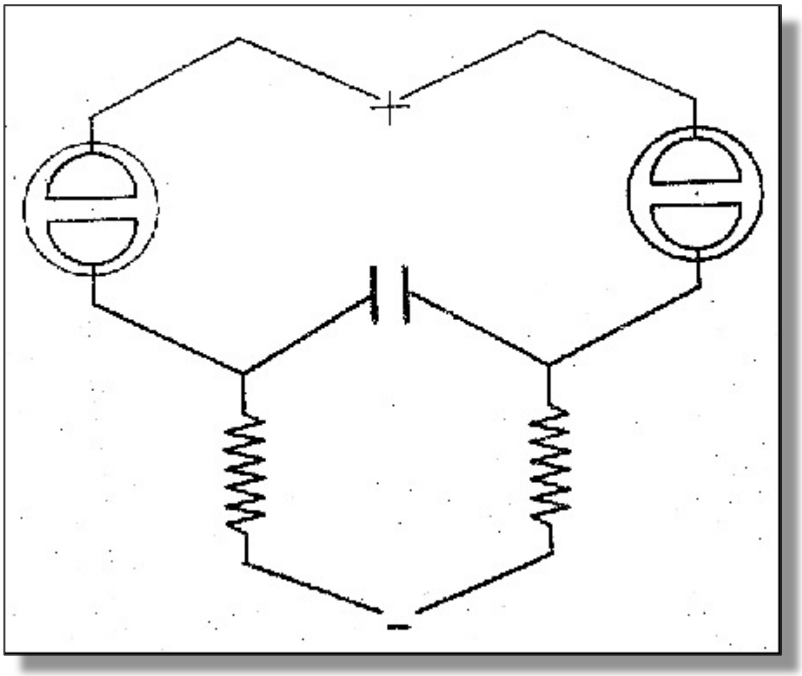


Mounted behind the diffuser is a hexagonal foam-core panel through which the neon lamps protrude. The lamps are pointed glass tubes about 1 inch long by ¼ inch in diameter, and are spaced about 1.5 inches apart in a uniform triangular grid that fills the panel. Pairs of insulated wires connect the anode and cathode of each lamp to an oscillator. The lamps act as nonlinear transducers, as they require a certain voltage to begin to glow, but continue to glow as voltage falls; when its voltage reaches a critical minimum (about 30% less than its ignition voltage), a lamp goes dark. This difference between ignition and extinction voltages is the nonlinear behavior that enables the lamps to trigger oscillations.

The lamps glow orange, neon's characteristic spectral signature. The amount of current through them governs their luminosity. Lamps that represent smaller a central places receive less current and so glow more dimly. They also flash more slowly. Both of these behaviors depend on the resistance and capacitance values of oscillator components and were designed to not vary within each level of the hierarchy, but a number of factors make them less than uniform.

Besides its two neon lamps, each oscillator has one capacitor and two resistors. The circuitry is the same for all of them and is very simple, as the following stylized wiring diagram, also taken from the device's 1969 documentation, shows.

Figure 5: Wiring diagram of a Hexagonia relaxation oscillator, one of 164 such circuits



In this circuit diagram, the neon lamps are depicted as circles, the resistors as jagged lines, and the capacitor as two parallel planes. The oscillator receives DC power at the nodes marked **+** and **–**. The two resistors and the capacitor are compactly wired together through holes in a printed circuit board. On the hexagonal panel, the paired lamps are diametrically opposed (i.e. radially symmetric) and alternate on/off states. Their opposition makes discerning which lamps are paired difficult, as does the fact that each pair of lamps blinks independently of the others. Because half the lamps are always on and half off, the amount of light the device emits stays roughly constant, which led the author to first call it *Steady State Microcosm*.

As mentioned above, higher-level lamps glow more brightly and blink more rapidly than lower level lamps. This behavior is controlled by the choice of oscillator circuit components. Lamp brightness is governed by the values of resistors (in Ohms). Higher resistance limits a lamp's current to produce a weaker glow. The product of that resistance and the associated capacitor's value (in Farads) determines how rapidly a pair of lamps blink. Consequently, higher-level circuits use smaller resistors and larger capacitors than lower-level circuits. Finding combinations of resistance and capacitance that made each level of the hierarchy perceptually distinct and pleasing required a series of tedious "breadboard" experiments punctuated by occasional jolts to the engineer, which did no lasting harm and helped keep him awake.

Central Place Statistics

Despite that Hexagonia's circuits run at constant rates, it does not seem to operate like clockwork. Its patterns of light and shade reconfigure rapidly and unpredictably. One reason is that the working parameters of its electrical components vary from their nominal specifications by as much as 20%. This means that it is quite unlikely that oscillators within the same order discharge at exactly the same rate. And, as individual oscillators are not synchronized, they produce instantaneous *macrostates* – visual configurations across the entire display – that exhibit randomness. As a result, the patterns of successor states defy prediction.

The number of Hexagonia's possible macrostates is finite but truly immense. As its oscillators operate independently and must be in one of two states at any given time, their total number of states is 2^n, where *n* is the number of oscillators. Taken together, the 164 oscillators have 2^{164}, or 2.34e49 possible states. Cycling through them at 20 state transitions per second (the fastest blink rate), it would take 3.71e40 years – a trillion times the age of the universe – to complete the repertoire. (As most of the oscillators run more slowly, enumerating all states would take even longer.)

However, this is not what happens. States flash by much more rapidly than 20 per second. Consider that the fastest lamp blinks 20 times per second, the six second-level lamps 10 per second, the 6 third-level lamps 5, and so on, simultaneously. Table 1 adds up these events.

Table 1: Activity Statistics for the Hexagonia Machine

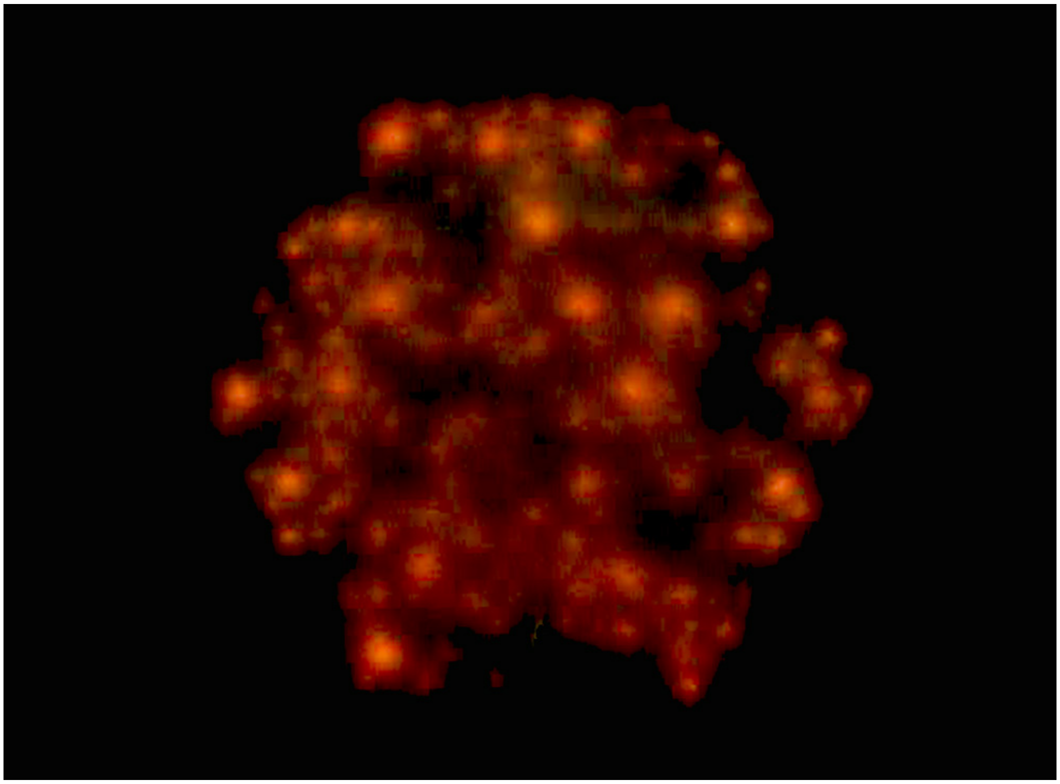
Level	**Number of Lamps**	**Flashes per Second**	**Events per Second**
1	1 *	20	20
2	6	10	60
3	6	5	30
4	24	2.5	60
5	72	1.25	90
6	218	.625	136
Total	327	1.2073 **	396

* The bright first-level lamp (at the center of the screen) proved to be distracting, and so was disabled to enhance perception of spatial patterns.
** Weighted average of flashes per second

With all levels of the hierarchy accounted for, the device generates almost 400 macrostates per second. Assuming that no two oscillators change state at exactly the same instant, all these states are mathematically and visually distinct, but are very fleeting. To capture each one, a video camera would need to run at least 14 times faster than its normal frame rate (30 FPS).

Figure 6 shows a frame from a VGA-resolution video of the device. It exhibits the temporal aliasing typical of too-slow shutter speeds. One clue that the image is aliased is that more than half the lamps seem to be glowing. Given that the human perceptual system resolves motion at less than 10 FPS, this snapshot is still less blurry than what the naked eye would apperceive.

Figure 6: Still frame from a VGA video of Hexagonia in operation



The Once and Future Hexagonia

When he created Hexagonia, the author was a graduate assistant in the Laboratory for Computer Graphics and Spatial Analysis (LCGSA) at Harvard's Graduate School of Design. Computer graphics was still in its infancy. The only digital graphic devices at his disposal were a mainframe computer's line printer and a pen plotter. While it would have been possible for him to program the mainframe to generate and print out sequences of hexagonal diagrams, and even to film them as an animation, it never occurred to him to do so. The line printer's coarse rendering would not have been adequate, debugging the code would have taken at least as long as it did to build the contraption, and no budget line was available at the university computer center to play with such whimsical notions.[4]

[4] A year or two later, LCGSA programmers equipped SYMAP—its flagship mapping program—with an algorithm that computed Theissen polygons, a generalized mapping of points to areas they influence.

Today, of course, Hexagonia could be implemented as an interactive application or a Web page with relative ease. A digital version would make slowing down and speeding up its pulsing a lot simpler than swapping resistors and capacitors in and out of the hardware. It could also alter the hierarchy from k=3 to k=4 or 7 and use color to highlight aspects of the simulation or paint the polygons like a thematic map. Its code could render more levels at once and even zoom through them infinitely. (Hexagonal hierarchies are, of course, fractals.)

A Hexagonia app could play on your smartphone. However, It would be difficult to algorithmically replicate the twinkly out-of-phase behavior of the original device—its quirks—that small, random mismatches in the properties of its electrical components create. Without that touch of stochasticity (which the programmer would have to deliberately incorporate), the display would run like clockwork and so deprive viewers of much of its original charm.

A digital version of Hexagonia was—and remains—a road not taken. But it could still be taken, should someone consider the effort to simulate a simulation of an elaboration of a theory to be sufficiently worthwhile.

So, any volunteers?

This document: http://spatial-effects.com/hexagonia/hexagonia.pdf

See Hexagonia: the Movie at http://spatial-effects.com/hexagonia/hexagonia-the-movie.mp4

Regular hexagonal grids are a special case of Theissen polygons. That feature could have produced central place maps, but they would have still been coarse and static.